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On the connection between $N = 2$ minimal string and $(1, n)$ bosonic minimal string

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Abstract: We study the scattering amplitudes in the $N = 2$ minimal string or equivalently in the $N = 4$ topological string on ALE spaces. We find an interesting connection between the tree level amplitudes of the $N = 2$ minimal string and those of the $(1, n)$ minimal bosonic string. In particular we show that the four and five-point functions of the $N = 2$ string can be directly rewritten in terms of those of the latter theory. This relation offers a map of physical states between these two string theories. Finally we propose a possible matrix model dual for the $N = 2$ minimal string in the light of this connection.

Keywords: 2D Gravity, Conformal Field Models in String Theory, Matrix Models, Topological Strings.

Contents

1. Introduction

A particularly interesting set of string theories is obtained by gauging $N = 2$ superconformal symmetry on the worldsheet. The resulting model is called the $N = 2$ string [2]. This string theory is very attractive since it has the highest amount of supersymmetry on the world-sheet with the positive critical dimension ($d = 4$). The simplest target space for the $N = 2$ string is the four dimensional flat spacetime with signature $(4, 0)$ or $(2, 2)$.

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This theory is referred as the critical $N = 2$ string. The theory with $(2, 2)$ signature can be solved exactly at least perturbatively. The physical spectrum of it contains a single massless scalar field whose dynamics is described by the mathematically beautiful theory of self-dual gravity [3].

Since the critical $N = 2$ string has rich mathematical structure, it is natural to hope that one can solve this string theory even non-perturbatively. Indeed in a similar setups like two dimensional bosonic or $N = 1$ type 0 string theory we know matrix model duals [4, 5] and can solve the theory non-perturbatively. These matrix models can be regarded as the open string theory of D0-branes [6] via the open-closed duality. Hence it is very likely that similar ideas can be applied to solve the $N = 2$ string.

In this paper we study the $N = 2$ string in a different background. The target space is the $N = 2$ coset space (Kazama-Suzuki model) $\mathrm{SL}(2, R)_n/\mathrm{U}(1) \times \mathrm{SU}(2)_n/\mathrm{U}(1)$. In contrast to the critical $N = 2$ string this theory has finite number of physical states. We will refer to it as the $N = 2$ minimal string [7, 8]. Again it is natural to expect that this model has the matrix model dual description as was true for the minimal bosonic [9] or type 0 string [10].

From higher dimensional string theory viewpoint, it is expected that the BPS sector of the Little String Theory (LST) (for recent review see [11, 12]) can be described by the $N = 2$ minimal string [13]. LST appears in various decoupling limits of string theories which contain NS5 branes or singularities. This theory was extensively studied using the holographically dual description [14]. The simplest of LSTs are $5 + 1$ dimensional theories with sixteen supercharges. They arise from the decoupling limit of k type IIA or type IIB $NS5$-branes in flat space. The holographic description of these theories is given by closed strings in the near horizon geometry of NS5 branes–the CHS background [15]. Unfortunately, string theory in this background is strongly coupled due to the presence of the linear dilaton. One way to avoid this problem is to consider the theory at a non-singular point in the moduli space. The simplest such configuration corresponds to NS5 branes distributed on a circle. In this case the CHS background gets deformed into [16]

$$\mathbb{R}^{5,1} \times \left(\frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2)}{\mathrm{U}(1)} \right) / \mathbb{Z}_k \,, \tag{1.1}$$

where the $\mathbb{Z}_k$ orbifolding ensures the R-charge integrality, i.e. imposes the GSO projection. We see that the non-trivial part of the background (1.1) coincides with that of the minimal $N = 2$ string.

It is instructive to consider the T-dual of the near horizon geometry of the NS5 branes. Under T-duality the CHS background is mapped to a singular ALE[1] space (the (1.1) is mapped to the resolved ALE). Then the physical states of the $N = 2$ string correspond to the deformation preserving the hyper-Kähler structure of the ALE space. In this sense the theory becomes topological and can be equivalently described as the $N = 4$ topological string ($N = 4$ TST) [18].

[1]The $N = 2$ string on the smooth ALE spaces defined by the orbifolds such as $\mathbb{C}^2/\mathbb{Z}_2$ was studied in [17]. See also the appendix A of the present paper for the analysis of physical states for the $\mathbb{C}^2/\mathbb{Z}_N$ orbifold.

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We study the tree level scattering amplitudes in the $N = 4$ topological string formulation using the equivalence between $N = 2$ string and $N = 4$ topological string theory [18]. We find that they are closely related to the amplitudes of the $(1, n)$ minimal bosonic string[2] by applying the recently found relation [21 − 24] between the correlation functions in the $\mathrm{SL}(2, R)$ (or H_3^+) WZW model and those in the bosonic Liouville theory. In particular we show that the four- and five-point correlation functions of the $N = 4$ topological string can be rewritten in terms of those of the $(1, n)$ minimal bosonic string. Moreover using this method we are able also to match some classes of higher point correlators. Even though our analysis is not exhaustive, it is possible that these two string theories are actually equivalent or at least closely related. In the end, based on this observation, we propose a candidate matrix model dual for the $N = 2$ minimal string.

The paper is organized as follows. In section 2 we give a brief review of the $N = 2$ minimal string and analyze its physical states. In section 3 we study the $N = 4$ topological string on ALE spaces which is equivalent to the $N = 2$ minimal string. In section 4 we compute the tree level scattering amplitudes and relate them to those of the $(1, n)$ minimal string. In section 5 we discuss our results and propose a possible matrix model dual for $N = 2$ minimal string.

2. $N = 2$ minimal string

2.1 Notations

In this subsection we set up notations. We are interested in string theories with $\mathrm{SU}(2)_n/\mathrm{U}(1) \times \mathrm{SL}(2, R)_n/\mathrm{U}(1)$ $N = 2$ superconformal matter[3] (we will often switch between $\mathrm{SL}(2)/\mathrm{U}(1)$ and the equivalent $N = 2$ Liouville descriptions). The $N = 2$ superconformal algebra reads

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \\
[L_m, G_r^{\pm}] &= \left(\frac{m}{2} - r\right) G^{\pm}_{m+r}\,, \\
[L_m, J_m] &= -nJ_{m+n}\,, \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}\,, \\
\{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0\,, \\
[J_n, G_r^{\pm}] &= \pm G^{\pm}_{r+n}\,, \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m,-n}\,,
\end{aligned} \tag{2.1}$$

[2]This connection was also speculated in [8] from the analysis of three point functions in the $N = 2$ string. We can also find an earlier work [19] which implies a relation between the $N = 2$ string and $c < 1$ bosonic string. It may also be closely related to the recent observation made in [20], where the equivalence between the $N = 2$ topological string on $AdS_3 \times S^3$ and the $(1, n)$ minimal string is argued from the cohomology analysis.

[3]Here n is the level of the supercoset WZW model. The bosonic part of the coset is given by $\mathrm{SU}(2)_{n-2}/\mathrm{U}(1) \times \mathrm{SL}(2, R)_{n+2}/\mathrm{U}(1)$.

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where c is the central charge, L_n are the Virasoro algebra generators, $G_r^{\pm}$ are the modes of the supercurrents ($r \in \mathbb{Z} + 1/2$ in the Neveu-Schwarz and $r \in \mathbb{Z}$ in the Ramond sectors) and J_n are the modes of the U(1) R-current.

Let us start by setting notation for the SU(2)/ U(1) coset. The central charge for this theory is

$$\hat{c}_{(su)} \equiv \frac{c}{3} = 1 - \frac{2}{n} \,. \tag{2.2}$$

The NS sector SU(2)/ U(1) superconformal primaries $\mathcal{V}$ are labeled by (j, m) quantum numbers, where

$$\begin{aligned} &2j \in \mathbb{Z},\ 0 \leq 2j \leq n-2\,, \\ &-j \leq m \leq j,\ j+m \in \mathbb{Z}\,. \end{aligned} \tag{2.3}$$

The dimension and the R-charge of the operator $\mathcal{V}$ can be expressed in terms of m and j as follows

$$\begin{aligned} \Delta_{(su)} &= \frac{j(j+1)}{n} - \frac{m^2}{n}\,, \\ R_{(su)} &= -\frac{2m}{n}\,. \end{aligned} \tag{2.4}$$

In order to unify the description of NS and R-sectors it is sometimes convenient to label operators in the SU(2)/ U(1) by three quantum numbers (j, m, s), where s is the spectral flow parameter. Not all (j, m, s) are independent, there is a following equivalence relation between them

$$(j, m, s) \sim (\frac{n-2}{2} - j, m + \frac{n-2}{2}, s+2)\,. \tag{2.5}$$

$s = 0, 2$ correspond to NS operator, while $s = \pm 1$ to Ramond sector operators. The dimension and the R-charge are

$$\begin{aligned} \Delta_{(su)} &= \frac{s^2}{8} + \frac{j(j+1)}{n} - \frac{(m+s/2)^2}{n}\,, \\ R_{(su)} &= \frac{s}{2} - \frac{2m+s}{n}\,. \end{aligned} \tag{2.6}$$

In the NS-sector for $|m| < j$ only $s = 0$ sector corresponds to superconformal primaries

$$\mathcal{V}^{(s=\pm 2)}_{j,m} \sim G^{\pm}_{-1/2}\mathcal{V}^{(s=0)}_{j,m\pm 1} \equiv G^{\pm}_{-1/2}\mathcal{V}_{j,m\pm 1}\,. \tag{2.7}$$

For $m = \pm j$, $\mathcal{V}^{(s=\pm 2)}_{j,\pm j}$ is actually primary. Indeed using (2.5) we find

$$\mathcal{V}^{(s=\pm 2)}_{j,\pm j} \sim \mathcal{V}_{\frac{n-2}{2}-j, \pm j \mp \frac{n-2}{2}}\,. \tag{2.8}$$

The spectral flow $s \to s \pm 2$ can be realized by using the spectral flow operator

$$\mathrm{SFO}^{\pm}_{(su)} = \mathcal{V}_{\frac{n-2}{2}, \mp\frac{n-2}{2}}\,. \tag{2.9}$$

This can be easily shown by using the SU(2) fusion rules and the relation (2.5).

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We would also like to comment on the relation of the $N = 2$ minimal model to the SU(2) WZW and the bosonic parafermions SU(2)/U(1). The $\mathrm{SU}(2)_{n-2}$ primary $\Psi_{j,m}$ can be expressed in terms of bosonic parafermion primary $V_{j,m}$ as follows[4]

$$\Psi_{j,m} = V_{j,m} e^{i\sqrt{\frac{2}{n-2}} m Y_3} . \tag{2.10}$$

The SU(2) currents can be expressed using parafermionic currents ψ_1 and $\psi_1^\dagger$

$$\begin{aligned} J^- &\sim \psi_1 \exp\left(-i\sqrt{\frac{2}{n-2}} Y_3\right) , \\ J^+ &\sim \psi_1^\dagger \exp\left(i\sqrt{\frac{2}{n-2}} Y_3\right) , \\ J^3 &\sim i\sqrt{\frac{n-2}{2}}\, \partial Y_3 . \end{aligned} \tag{2.11}$$

The spectral flowed primary operator is defined

$$\Psi^w_{j,m} = V_{j,m} e^{i\sqrt{\frac{2}{n-2}}(m+\frac{n-2}{2}w) Y_3} . \tag{2.12}$$

This has the eigenvalue $J_3 = m + \frac{n-2}{2} w$ and the conformal dimension $\Delta = \frac{j(j+1)}{n} + mw + \frac{n-2}{4} w^2$. Note that for the SU(2), because of the Weyl identification $m \sim m + (n-2)$ only spectral flow by $w = \pm 1$ are independent. Using the identification of the quantum numbers

$$(j, m, w) \sim \left(\frac{n-2}{2} - j, \frac{n-2}{2} + m, w-1\right) , \tag{2.13}$$

and the fact that the operator $\Psi_{j,m}$ is SU(2) current primary only for $-j \le m \le j$ we find that

$$\Psi^{w=\pm 1}_{j,m} = (J^\pm_{-1})^{(j\pm m)} \Psi^{w=0}_{\frac{n-2}{2}-j, \pm\frac{n-2}{2}\mp j} . \tag{2.14}$$

One can rewrite the $N = 2$ minimal model operators using the bosonic parafermions (see [25] and the formula (4.47) and (4.51) in [13])

$$\mathcal{V}^s_{j,m} = V_{j,m} e^{i\frac{-2m + s\frac{n-2}{2}}{\sqrt{n(n-2)}} Y} \equiv V_{j,m} e^{i\alpha_{m,s} Y} . \tag{2.15}$$

Field Y is essentially the bosonization of the R-current. The $N = 2$ supercurrents can also be rewritten[5] in term of ψ_1 and $\psi_1^\dagger$

$$\begin{aligned} G^+ &\sim \psi_1 \exp\left(i\sqrt{\frac{n}{n-2}} Y\right) , \\ G^- &\sim \psi_1^\dagger \exp\left(-i\sqrt{\frac{n}{n-2}} Y\right) , \\ J_R &\sim i\sqrt{\frac{n-2}{n}}\, \partial Y . \end{aligned} \tag{2.16}$$

[4] We normalized the boson Y_3 such that $Y_3(z) Y_3(0) \sim -\log z$.

[5] Again we normalized the boson Y such that $Y(z) Y(0) \sim -\log z$.

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Using these relations one can explicitly check (2.7). Now we are in position to express a general correlator in the $N=2$ minimal model in terms of correlators of SU(2) WZW

$$\left\langle \mathcal{V}^{s_1}_{j_1,m_1},\cdots,\mathcal{V}^{s_N}_{j_N,m_N}\right\rangle = \left\langle V_{j_1,m_1},\cdots V_{j_N,m_N}\right\rangle \left\langle e^{i\alpha_{m_1,s_1}Y}\cdots e^{i\alpha_{m_N,s_N}Y}\right\rangle . \tag{2.17}$$

Note that the R-charge conservation in the above formula requires

$$\sum m_i - \frac{n-2}{4}s_i = 0\,, \tag{2.18}$$

which means that we cannot lift the amplitude of bosonic parafermions to the amplitude of SU(2) primaries, unless $\sum s_i = 0$. But it can be lifted into an amplitude involving $\Psi^w_{j,m}$. Indeed if we have

$$w_i = -s_i/2\,, \tag{2.19}$$

the sum rule for Y_3 momentum will be satisfied and we will get

$$\langle \mathcal{V}^{s_1}_{j_1,m_1},\ldots,\mathcal{V}^{s_N}_{j_N,m_N}\rangle = \frac{\langle \Psi^{w_1}_{j_1,m_1},\ldots,\Psi^{w_N}_{j_N,m_N}\rangle\langle e^{i\alpha_{m_1,s_1}Y}\cdots e^{i\alpha_{m_N,s_N}Y}\rangle}{\langle e^{i\beta_{m_1,s_1}Y_3}\cdots e^{i\beta_{m_N,s_N}Y_3}\rangle}\,, \tag{2.20}$$

where (cr. (2.12))

$$\beta_{m,s} = \sqrt{\frac{2}{n-2}}\left(m - \frac{n-2}{4}s\right). \tag{2.21}$$

The case of chiral primary field ($m=-j$) is a bit different since it can be represented as both $s=0$ and $s=2$ operator (2.8).

It is straightforward to compute the free field parts in the correlation function (2.20) as follows

$$\frac{\langle\prod_{i=1}^N e^{i\alpha_{m_i,s_i}Y(z_i)}\rangle}{\langle\prod_{i=1}^N e^{i\beta_{m_i,s_i}Y_3(z_i)}\rangle} = \prod_{1\le i<j\le N}(z_i-z_j)^{-\frac{2}{n}(m_i-\frac{n-2}{4}s_i)(m_j-\frac{n-2}{4}s_j)}\,. \tag{2.22}$$

The discussion for $\mathrm{SL}(2,R)/\,\mathrm{U}(1)$ case is essentially the same. Here the superconformal operators $\mathcal{V}'$ are labeled by quantum numbers (h,m,s), where h is the $\mathrm{SL}(2,R)$ spin, m is J_3 quantum number related to the R-charge and s is the spectral flow parameter. The formulae for the central charge, the dimension and the R-charge of superconformal operators can be obtained from the corresponding formulae for $\mathrm{SU}(2)/\,\mathrm{U}(1)$ by taking $j\to -h$, $n\to -n$:

$$\begin{aligned}
\hat{c}_{(sl)} &= 1+\frac{2}{n}\,,\\
\Delta_{(sl)} &= \frac{s^2}{8}-\frac{h(h-1)}{n}+\frac{(m+s/2)^2}{n}\,,\\
R_{(sl)} &= \frac{s}{2}+\frac{2m+s}{n}\,.
\end{aligned} \tag{2.23}$$

But there are also important differences. In particular, the formulae (2.5), (2.7), (2.8) and (2.14), which can be rewritten[6] as

$$(h,m,s)\sim\left(\frac{n+2}{2}-h, m-\frac{n+2}{2}, s+2\right),$$

[6] $\Phi^w_{h,m}$ denotes the spectral flowed primary [26] of $\mathrm{SL}(2,R)_{n+2}$ WZW model. It has the eigenvalue $J_3 = m - \frac{n+2}{2}w$ and the conformal dimension $\Delta = -\frac{h(h-1)}{n} + mw - \frac{n+2}{4}w^2$.

$$
\begin{aligned}
\mathcal{V}'^{s=\pm 2}_{h,m} &= G^{\pm}_{-1/2}\mathcal{V}'^{s=0}_{h,m\pm 1}\,, \\
\mathcal{V}'^{s=\pm 2}_{h,\mp h} &\sim \mathcal{V}'^{s=0}_{\frac{n+2}{2}-h\,,\mp h\pm\frac{n+2}{2}}\,, \\
\Phi^{w=\pm 1}_{h,m} &= (J^{\pm}_{-1})^{m-h}\;\Phi^{w=0}_{\frac{n+2}{2}-h,\mp\frac{n+2}{2}\pm h}\,.
\end{aligned} \tag{2.24}
$$

only apply to the operators corresponding to the discrete ($h \in \mathbb{R}$, $h-m \in \mathbb{Z}$ or $h+m \in \mathbb{Z}$) (chiral or anti-chiral) and degenerate representations ($2h \in -\mathbb{Z}_{>0}$, $-|h| \leq m \leq |h|$) of SL(2). The spectral flow operator in the SL(2)/ U(1) is

$$
\mathrm{SFO}^{\pm}_{(sl)} = \mathcal{V}'_{\frac{n+2}{2},\pm\frac{n+2}{2}}\,. \tag{2.25}
$$

We would like to derive the analog of the formula[7] (2.20) for SL(2)/ U(1)

$$
\langle \mathcal{V}'^{s_1}_{h_1,m_1},\cdots,\mathcal{V}'^{s_N}_{h_N,m_N}\rangle = \frac{\langle \Phi^{w_1}_{h_1,m_1},\cdots,\Phi^{w_N}_{h_N,m_N}\rangle\langle e^{i\alpha'_{m_1,s_1}X}\cdots e^{i\alpha'_{m_N,s_N}X}\rangle}{\langle e^{\beta'_{m_1,s_1}X_3}\cdots e^{\beta'_{m_N,s_N}X_3}\rangle}\,, \tag{2.26}
$$

where

$$
\begin{aligned}
\beta'_{m,s} &= \sqrt{\frac{2}{n+2}}\left(m+\frac{n+2}{4}s\right), \\
\alpha'_{m,s} &= -\frac{2m+s\frac{n+2}{2}}{\sqrt{n(n+2)}}\,, \\
w &= -\frac{s}{2}\,.
\end{aligned} \tag{2.27}
$$

Finally we can also find the following formula analogous to (2.22)

$$
\frac{\langle \prod_{i=1}^{N} e^{i\alpha'_{m_i,s_i}X(z_i)}\rangle}{\langle \prod_{i=1}^{N} e^{i\beta'_{m_i,s_i}X_3(z_i)}\rangle} = \prod_{1\leq i<j\leq N}(z_i-z_j)^{\frac{2}{n}(m_i+\frac{n+2}{4}s_i)(m_j+\frac{n+2}{4}s_j)}\,. \tag{2.28}
$$

2.2 $N = 2$ minimal string theory

The $N = 2$ minimal string theory is defined by coupling the $N = 2$ minimal matter to the $N = 2$ Liouville theory [27, 28], and gauging the world-sheet $N = 2$ superconformal symmetry. The $N = 2$ minimal model is equivalent to the $N = 2$ coset SU(2)/ U(1), while the $N = 2$ Liouville theory is T-dual to the $N = 2$ coset SL$(2, R)$/ U(1) via the supersymmetric FZZ duality [16, 29]. Therefore the target space of the $N = 2$ minimal string theory is given by the following product of two $N = 2$ superconformal cosets [30, 19] (as we will see below this product actually enjoys enhanced $N = 4$ superconformal symmetry on the world-sheet)

$$
\left[\frac{\mathrm{SL}(2,R)_n}{\mathrm{U}(1)}\times\frac{\mathrm{SU}(2)_n}{\mathrm{U}(1)}\right]/\mathbb{Z}_n\,, \tag{2.29}
$$

with the total central charge

$$
\hat{c}_{tot} = 2\,, \tag{2.30}
$$

[7] In the same way as before we normalized the free boson X_3 and X such that $X(z)X(0) \sim X_3(z)X_3(0) \sim -\log z$.

where the $\mathbb{Z}_n$ orbifold in (2.29) ensures integral R-charges of states as required by the modular invariance of the $N = 2$ string. The $N = 2$ string contains the usual (b, c) conformal ghosts, two pairs of superconformal ghosts $(\beta^{\pm}, \gamma^{\mp})$ and an additional $(\tilde{b}, \tilde{c})$ fermionic ghost system, which arises from gauging the R-current. One can check that the total central charge of the $N = 2$ ghost system is

$$\hat{c}_{gh} = -2\,, \tag{2.31}$$

which precisely matches the the central charge of the matter sector. It is convenient to bosonize the β, γ system as follows

$$\beta^{\pm} \sim e^{-\phi_{\mp}} \partial \xi^{\pm}; \qquad \gamma^{\pm} \sim \eta^{\pm} e^{\phi_{\pm}}\,. \tag{2.32}$$

The physical states of the $N = 2$ string theory are elements of cohomology groups of the BRST operator

$$Q_{BRST} = \frac{1}{2\pi i} \oint dz\, j_{BRST}\,, \tag{2.33}$$

where the BRST current takes the form

$$\begin{aligned} j_{BRST} &= cT + \eta_- e^{\phi_-} G^+ + \eta_+ e^{\phi_+} G^- + \tilde{c} J^m + \\ &\quad \frac{1}{2}[cT^{gh} + \eta_- e^{\phi_-} G^+_{gh} + \eta_+ e^{\phi_+} G^-_{gh} + \tilde{c} J^{gh}]\,. \end{aligned} \tag{2.34}$$

The BRST current has non-singular OPE with the two picture number currents

$$j_{\pi+} = -\eta^+ \xi^- - \partial \phi_+; \qquad j_{\pi-} = -\eta^- \xi^+ - \partial \phi_-\,, \tag{2.35}$$

and the ghost number current

$$j_{gh} = -bc - \tilde{b}\tilde{c} + \eta^+ \xi^- + \eta^- \xi^+\,. \tag{2.36}$$

Hence the corresponding cohomology groups are labeled by the ghost number and the picture numbers (Π_+, Π_-). One can define two picture raising operators

$$PCO^{\pm} = \{Q, \xi^{\pm}\} = c\partial\xi^{\pm} + e^{\phi^{\mp}}(G^{\pm} - 2\eta^{\pm} e^{\phi^{\pm}} b \pm 2\partial(\eta^{\pm} e^{\phi^{\pm}})\tilde{b} \pm \eta^{\pm} e^{\phi^{\pm}} \partial \tilde{b})\,. \tag{2.37}$$

It is known [31] that unlike in the $N = 1$ superstrings, the picture raising operators (2.37) are not isomorphisms of the cohomology groups at different pictures, which in general complicates analysis of cohomologies.

Let us now discuss the cohomologies of the BRST operator at lower pictures and ghost number one. We will follow closely the discussion in [7]. The BRST invariant operator in the standard $(-1, -1)$ picture can be written as [8]

$$V^{(-1,-1)}_{j,m} = c\mathcal{V}_{j,m} \mathcal{V}'_{h=-j,m} e^{-\phi_+} e^{-\phi_-}. \tag{2.38}$$

[8] In this section we write only the chiral part of the operators to keep the expressions simple.

The operator $\mathcal{V}$ and $\mathcal{V}'$ should be primaries of the corresponding $\mathcal{N}=2$ algebras, in order for $\mathcal{O}$ to be BRST closed. This operator, for generic m, has images in the lower pictures. This can be shown using (2.37). The result is

$$V_{j,m}^{(0,-1)} = cG^{-}_{-\frac{1}{2}}(\mathcal{V}_{j,m}\mathcal{V}'_{-j,m})e^{-\phi_-}. \tag{2.39}$$

In general the BRST cohomologies in this picture have the following form

$$\tilde{\mathcal{O}} = ce^{-\phi_-}\tilde{\mathcal{V}}\tilde{\mathcal{V}}', \tag{2.40}$$

where $\tilde{\mathcal{V}}\tilde{\mathcal{V}}'$ satisfies

$$\begin{aligned} G_r^+(\tilde{\mathcal{V}}\tilde{\mathcal{V}}') &= 0\,, \\ G_{r-1}^-(\tilde{\mathcal{V}}\tilde{\mathcal{V}}') &= 0, \quad r>0\,, \\ \Delta(\tilde{\mathcal{V}}\tilde{\mathcal{V}}') &= \frac{1}{2}\,, \\ q(\tilde{\mathcal{V}}\tilde{\mathcal{V}}') &= -1\,. \end{aligned} \tag{2.41}$$

These conditions are indeed satisfied for the $V_{j,m}^{(0,-1)}$, but it is easy to see that in this picture there are additional BRST invariant operators corresponding to the fields which are anti-chiral in $\mathrm{SU}(2)/\mathrm{U}(1)$ and $\mathrm{SL}(2,R)/\mathrm{U}(1)$ separately

$$\phi_h^{(0,-1)} = ce^{-\phi_-}\mathcal{V}_{\frac{n}{2}-h,\frac{n}{2}-h}\mathcal{V}'_{h,-h}\,. \tag{2.42}$$

It is clear that this operator is not of the form (2.39), hence does not have preimage in the $(-1,-1)$ picture. In this paper we will be mainly interested in the physical operators of this type. Vertex operators based on chiral fields in the $\mathrm{SU}(2)/\mathrm{U}(1)$ and $\mathrm{SL}(2)/\mathrm{U}(1)$ can be constructed in the similar manner. These vertex operators live naturally in the $(-1,0)$ picture

$$\phi_h^{(-1,0)} = ce^{-\phi_+}\mathcal{V}_{\frac{n}{2}-h,-\frac{n}{2}+h}\mathcal{V}'_{h,h}\,. \tag{2.43}$$

The operators in the $(0,-1)$ and $(-1,0)$ pictures are related via spectral flow, which can be realized using the following BRST invariant operators

$$(S^{\pm})^2 = e^{\pm\tilde{b}c}e^{\phi_\pm}e^{-\phi_\mp}\,\mathrm{SFO}^{\mp}_{(su)}\,\mathrm{SFO}^{\mp}_{(sl)}\,. \tag{2.44}$$

Indeed using (2.44) one can immediately see that

$$\phi^{(0,-1)}_{\frac{n+2}{2}-h} = (S^+)^2\phi_h^{(-1,0)}\,. \tag{2.45}$$

In order to compute correlation functions one also needs the integrated form of these operators in $(-1,0)$ and $(0,0)$ picture

$$\phi_{h,int}^{(0,-1)} = \textstyle\int e^{-\phi_-}\mathcal{V}_{\frac{n}{2}-h,\frac{k}{2}-h}\mathcal{V}'_{h,-h}\,,\ \phi_{h,int}^{(-1,0)} = \displaystyle\int e^{-\phi_+}\mathcal{V}_{\frac{n}{2}-h,-\frac{n}{2}+h}\mathcal{V}'_{h,h}\,. \tag{2.46}$$

Applying (2.37) one can easily find the form of these operators in the $(0,0)$ picture

$$\begin{aligned} PCO^+\phi_{h,int}^{(0,-1)} &= \int c\partial\xi^+e^{-\phi_-}\mathcal{V}_{\frac{n}{2}-h,\frac{n}{2}-h}\mathcal{V}'_{h,-h} + \int G^+_{-1/2}(\mathcal{V}_{\frac{n}{2}-h,\frac{n}{2}-h}\mathcal{V}'_{h,-h})\,, \\ PCO^-\phi_{h,int}^{(-1,0)} &= \int c\partial\xi^-e^{-\phi_+}\mathcal{V}_{\frac{n}{2}-h,-\frac{n}{2}+h}\mathcal{V}'_{h,h} + \int G^-_{-1/2}(\mathcal{V}_{\frac{n}{2}-h,-\frac{n}{2}+h}\mathcal{V}'_{h,h})\,. \end{aligned} \tag{2.47}$$

As we will see below it is sufficient for our purposes to retain only the second term in these expressions. Restoring the $\bar{z}$ dependence we find

$$
\begin{aligned}
PCO^{+}P\bar{C}O^{+}\phi_{int}^{(0,-1)} &= \int d^2z G^{+}_{-1/2}\bar{G}^{+}_{-1/2}(\mathcal{V}_{\frac{n}{2}-h,\frac{n}{2}-h}\mathcal{V}'_{h,-h}) + \cdots\,, \\
PCO^{-}P\bar{C}O^{-}\phi_{int}^{(-1,0)} &= \int d^2z G^{-}_{-1/2}\bar{G}^{-}_{-1/2}(\mathcal{V}_{\frac{n}{2}-h,-\frac{n}{2}+h}\mathcal{V}'_{h,h}) + \cdots\,.
\end{aligned}
\tag{2.48}
$$

It is also interesting to note that the operators $PCO^{\pm}$ may map the BRST non-trivial operators into the BRST trivial ones. This statement in the free $N=2$ string translates into the statement that picture changing acts as an isomorphism of the BRST cohomology groups only for the states with non-zero momentum. The simplest example of such phenomenon is the operator

$$
\mathcal{O} = ce^{-\phi_-}e^{-\phi_+}\mathbf{1}\,. \tag{2.49}
$$

This operator is mapped into zero by the action of both $PCO^{\pm}$. In general BRST non-trivial operators of the type

$$
\mathcal{O} = ce^{-\phi_-}e^{-\phi_+}\mathcal{V}\mathcal{V}'\,, \tag{2.50}
$$

where $\mathcal{V}$ and $\mathcal{V}'$ are (anti-)chiral primaries of $\mathrm{SU}(2)/\,\mathrm{U}(1)$ and $\mathrm{SL}(2)/\,\mathrm{U}(1)$ respectively are mapped into zero by the action of PCO^{+} (PCO^{-}).

Finally, to check the consistency of the string theory we need to find a modular invariant partition function. We worked this out in the appendix C.

3. Equivalent description as $N=4$ topological string on ALE spaces

3.1 Exhibiting the $N=4$ structure of $\mathrm{SU}(2)/\,\mathrm{U}(1)\times\mathrm{SL}(2)/\,\mathrm{U}(1)$ $N=2$ theory

As noted in [32, 18] any $N=2$ superconformal theory with the central charge

$$
\hat{c} = 2\,, \tag{3.1}
$$

automatically has $N=4$ superconformal symmetry. The $N=4$ superconformal algebra is defined by the energy-momentum tensor T, the four supercurrents $(G^+,\tilde{G}^+,G^-,\tilde{G}^-)$ and the SU(2) currents (J^{++},J^{--},J^3). J_3 is identified with the R-current in the $N=2$ subalgebra. $(G^+,\tilde{G}^+)$ have J_3 charge $+1$ while $(G^-,\tilde{G}^-)$ have the -1 charge.

For the particular value (3.1) of the central charge the spectral flow operators $\mathrm{SFO}^{\pm}$ have the R-charge ± 2 and dimension one and hence can serve[9] as J^{++} and J^{--} currents of the $N=4$. In the case at hand

$$
J^{\pm\pm} = \mathrm{SFO}^{\pm} \equiv \mathrm{SFO}^{\pm}_{(su)}\,\mathrm{SFO}^{\pm}_{(sl)}\,. \tag{3.2}
$$

Also $\tilde{G}^+$ and $\tilde{G}^-$ can be found as

$$
\tilde{G}^+_r = [J_0^{++},G^-_r]\,,\qquad \tilde{G}^-_r = [J_0^{--},G^+_r]. \tag{3.3}
$$

[9] We work in the notations of [18].

It is well known that the deformations of the theory which respect the $N=2$ structure are in one to one correspondence with chiral operators of R-charge 1 and dimension 1/2. A deformation by an operator V respects the $N=4$ structure[10] iff [18]

$$G^{+}_{-1/2}V=0,\ \tilde{G}^{+}_{-1/2}V=0,\ J_0^{++}V=0,\ J_0^{--}V \text{ is anti}-\text{chiral}\,. \tag{3.4}$$

It is instructive to check that the deformations of the type

$$V_h=\mathcal{V}_{\frac{n}{2}-h,-\frac{n}{2}+h}\mathcal{V}'_{h,h}\,, \tag{3.5}$$

respect the $N=4$ structure. The first of the (3.4) relations is obviously satisfied, since both SU(2)/ U(1) and SL(2)/ U(1) operators are chiral primaries. Using (2.5) one can see also that[11]

$$J_0^{--}V_h=\bar{V}_{\frac{n+2}{2}-h}\,, \tag{3.6}$$

hence the fourth relation is also correct. It is not hard to show that the second and third relations also hold. We notice that the V_h deformations are in one to one correspondence with the accidental cohomologies $\phi_h^{(-1,0)}$ of the $N=2$ string (2.42). One can similarly check that the deformations which correspond to the $V_{j,m}^{(-1,0)}$ (see (2.39))

$$G^{+}_{-1/2}(\mathcal{V}_{j,m},\mathcal{V}'_{h=-j,m})\equiv G^{+}_{-1/2}V_{j,m}\,, \tag{3.7}$$

also respect the $N=4$ structure.

3.2 Geometrical interpretation

The target space defined by the $\hat{c}=2$ coset space (2.29) can also be regarded as the resolved A_{n-1} singularity [19, 16][12]

$$x^n+y^2+z^2=\mu_{sl}\,. \tag{3.8}$$

The deformation parameter μ_{sl} is equal to the $N=2$ cosmological constant in the dual $N=2$ Liouville theory. This background can be regarded as regularized CHS geometry [15, 30]. Indeed it is well-known that it is T-dual to the near horizon geometry of n NS5-branes situated on a circle of radius $r_0\sim(\mu_{sl})^{\frac{1}{n}}$ [16].

The Kähler and complex structure deformations of the ALE space are described by the (c,c), (a,a), (a,c) and (c,a) rings of the SCFT (2.29). There are $4(n-1)$ such deformations for the A_{n-1} ALE space–two complex structure and two Kähler deformations for each 2-cycle.

These deformations are described in the SCFT by the interaction terms on the world-sheet (we show only left-moving part)

$$\sum_{i=1}^{n-1}\left(t_{1L}^{(i)}\int G^{-}_{-1/2}V^{(i)}+t_{2L}^{(i)}\int \tilde{G}^{-}_{-1/2}V^{(i)}\right)\,, \tag{3.9}$$

[10]The third and fourth conditions are always true in the unitary SCFT when V is a chiral field with the R-charge +1. Here we are not assuming the unitarity of the underlining SCFT.

[11]We define the complex conjugate $\bar{V}_h$ of V_h by $\bar{V}_h=\mathcal{V}_{\frac{n}{2}-h,\frac{n}{2}-h}\mathcal{V}'_{h,-h}$, flipping the sign of J_3.

[12]To make the connection with the coset CFT (2.29) clearer we can express it in the homogeneous coordinates as $x^n+y^2+z^2-\mu_{sl}\ u^{-n}=0$. The $N=2$ Landau-Ginzburg model for $W=x^n$ and $W=u^{-n}$ is equivalent to the SU and SL part of the coset [19].

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where the second term can be written as $G^+\bar{V}$[13]. By adding the right-moving sector, we have four parameters[14] $t_{1L}^{(i)}, t_{2L}^{(i)}, t_{1R}^{(i)}$ and $t_{2R}^{(i)}$ for each $i = 1, 2, \ldots, n-1$. The four combinations $t_{1L}^{(i)}t_{1R}^{(i)}, t_{1L}^{(i)}t_{2R}^{(i)}, t_{2L}^{(i)}t_{1R}^{(i)}$ and $t_{2L}^{(i)}t_{2R}^{(i)}$ are the moduli corresponding to the $i-$th 2-cycle.

As we have seen in the previous subsection there are $n-1$ operators (again we show only the left moving part) of the type (3.5) in our model (2.29)

$$V_h \qquad (2h-1 = 1, 2, \ldots, n-1)\,. \tag{3.10}$$

After including the right movers we get exactly $4(n-1)$ operators. These operators are responsible for the Kähler and complex structure deformations of the ALE space. Notice that this finite number of allowed states come from the familiar bound (2.3) of j in the $\mathrm{SU}(2)_{n-2}$ model and it is also consistent with the unitarity bound of $\mathrm{SL}(2,R)_{n+2}$ model [26]. Similarly, we can check that $n-1$ twisted sector states exist in the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ which satisfy (3.4) as we show in the appendix A (see also [17]).

In addition to the states (3.10) there exist other deformations (3.7) which come from $(-1,-1)$ picture states. At present we do not have any clear geometrical interpretation of them. As we cannot find any such states as (3.7) in the orbifold case $\mathbb{C}^2/\mathbb{Z}_n$, the number of this type of states may not be conserved in the geometrical deformations of the theory. Thus we will concentrate on the states (3.10) in the rest of this paper.

3.3 Definition of $N = 4$ topological string

Sometimes it is useful to employ the $N = 4$ topological string description which is known to be equivalent to the $N = 2$ string [18].

The $N = 4$ topological string is defined as follows. Consider a $\hat{c} = 2$ $N = 4$ SCFT and perform the usual topological twist $T \to T + \frac{1}{2}\partial J^3$ [34, 18]. After the twist the operators G^+ and $\tilde{G}^+$ have the conformal dimension $\Delta = 1$, while G^- and $\tilde{G}^-$ have $\Delta = 2$. Since the former ones satisfy $(G_0^+)^2 = (\tilde{G}_0^+)^2 = \{G_0^+, \tilde{G}_0^+\} = 0$, they behave like BRST operators.

The physical states have R-charge one and are the top components of an SU(2) doublet, whose bottom components are anti-chiral with $R = -1$. They satisfy

$$G_0^+ V = \tilde{G}_0^+ V = 0\,, \tag{3.11}$$

and are subject to equivalence relation $V \sim V + G_0^+\tilde{G}_0^+\chi$. Conditions (3.11) are equivalent to (3.4) before we take the topological twist. Hence the physical states of the $N = 4$ string are in one to one correspondence with the $(-1,0)$ picture states of the $N = 2$ string.

3.4 Physical states in $N = 4$ topological string

Here we summarize physical states in $N = 4$ topological string on the ALE spaces. We present the vertex operators from the NS-sector viewpoint before the topological twisting.

In this paper we are mainly interested in the operators of the type (3.5), which are separately chiral in the SU(2)/U(1) and SL(2)/U(1) sectors of the theory. As we have

[13] Indeed using (3.6) and (3.3) one can show that $G^+\bar{V}_h = \tilde{G}^- V_{\frac{n+2}{2}-h}$.

[14] In the notation in [18], [33] these correspond to the twistor variables $u_{1L}, u_{2L}, u_{1R}, u_{2R}$.

seen above these operators correspond to the special cohomologies of the $N = 2$ string in the $(-1, 0)$ picture. Then we find $(n-1)$ physical states (3.5) corresponding to the deformations which respect the $N = 4$ structure (we show only left-moving index m and omit $\bar{m}$)

$$V_h = \mathcal{V}_{n/2-h,-n/2+h}\mathcal{V'}_{h,h}\,, \tag{3.12}$$

where h runs over $n-1$ half integers $h = 1, 3/2, \ldots, n/2$.

One also finds $(n-1)$ antichiral states, which correspond to $(0,-1)$ picture special cohomologies of the $N = 2$ string

$$J_0^{--}V_h = \mathcal{V}^{(s=-2)}_{n/2-h,-n/2+h}\mathcal{V'}^{(s=-2)}_{h,h} = \mathcal{V}_{h-1,h-1}\mathcal{V'}_{n/2-h+1,-n/2+h-1}\,, \tag{3.13}$$

where to get the last line we used (2.5) and (2.24). We will see below that they are needed to define the correlators of the $N = 4$ TST. These chiral and anti-chiral states correspond to the two types of the deformations $t_1^{(i)}$ and $t_2^{(i)}$ in (3.9), respectively.

Finally we would like to mention again that there are other physical states (3.7) whose geometrical meaning is not clear. It would be an interesting future problem to study them further. See [8] for the tree level scattering amplitudes for these states.

4. Scattering amplitudes in the $N = 4$ topological string on ALE

4.1 $N = 4$ topological string amplitudes

Let us consider $N \geq 4$ particle scattering amplitudes A_N in $N = 4$ topological string at tree level. Following Berkovits and Vafa [18] it is defined by

$$A_N = \left\langle \left[\int d^2z_1 J_L^{--} J_R^{--} V_1(z_1)\right] \; V_2(z_2)\; V_3(z_3)\; V_4(z_4) \prod_{a=5}^{N} \int d^2z_a G_L^- G_R^- V_a(z_a)\right\rangle_{TST} . \tag{4.1}$$

The physical states V_i have the R-charge $R = 1$ and the topological dimension 0. The total R-charge is $\sum_{i=1}^N q_i - 2 - (N-4) = 2 = \hat{c}$, which shows the charge conservation violation (or ghost number anomaly) familiar in topological string.

The integrated vertex operators $\int d^2z_a G_L^- G_R^- V_a(z_a)$ correspond to one of the four deformations $t_{1L}t_{1R}$ in (3.9). The other three vertex operators can be found by replacing either or both of $G^-_{L,R}$ with $\tilde{G}^-_{L,R}$.

To compute the correlation function (4.1) we need to rewrite it in terms of untwisted correlators. Then we can compute it using the results in conformal field theory. This can be done by inserting the spectral flow operator J^{--}. We insert it at the point $z = z_2$, though it can be inserted at any point (refer to e.g. [35]). Then the (4.1) can be written as

$$\begin{aligned} A_N = |z_2 - z_3|^2|z_2 - z_4|^2\Big\langle &\left[\int d^2z_1 |z_1 - z_2|^{-2} J_{0L}^{--} J_{0R}^{--} V_1(z_1)\right] \; [J_{0L}^{--} J_{0R}^{--} V_2(z_2)] \\ &\cdot V_3(z_3)\; V_4(z_4) \prod_{a=5}^{N} \int d^2z_a G^-_{-1/2L} G^-_{-1/2R} V_a(z_a)\Big\rangle_{untwitsed}. \end{aligned} \tag{4.2}$$

This formula can be easily understood from the point of view of the $N = 2$ string. Indeed using (2.43), (2.46) and (2.48) we find

$$A_N = \langle \phi^{(0,-1)}_{1,int}\phi^{(0,-1)}_2(z_2)\phi^{(-1,0)}_3(z_3)\phi^{(-1,0)}_4(z_4)\prod_{a=5}^{N}|PCO^-|^2\phi^{(-1,0)}_{a,int}\rangle\,, \tag{4.3}$$

where

$$\begin{aligned}\phi^{(-1,0)} &= ce^{-\phi_+}V\,,\\ \phi^{(0,-1)} &= (S^+)^2\phi^{(-1,0)}\,,\end{aligned} \tag{4.4}$$

and the operator ϕ_{int} is defined by formulae (2.46) and (2.47). We see that only the first term in (2.48) contributes to the correlator (4.3) due to anomalous conservation of $\partial\phi_\pm$ currents.

From the physical string theory viewpoint, the amplitudes A_N (4.2) or (4.3) compute the coupling of 4 RR-fields and $N-4$ NSNS-fields in the Little String Theory[15]. In the case of the four point functions this was discussed in [13].

We can identify the vertex V_a and the spectral flowed one $J^{--}_{0L}J^{--}_{0R}V_a$ in (4.2) with the physical states (3.12) and (3.13). In particular, we will find convenient the following expressions which are equivalent to (3.12) and (3.13) via the identity (2.8) (we suppress the right-moving part)

$$\begin{aligned}V_h &= \mathcal{V}^{(s=2)}_{h-1,h-1}\mathcal{V}'^{(s=0)}_{h,h}\,,\\ J^{--}_{0L}J^{--}_{0R}V_h &= \mathcal{V}^{(s=0)}_{h-1,h-1}\mathcal{V}'^{(s=-2)}_{h,h}\,.\end{aligned} \tag{4.5}$$

We also find another equivalent representation using (2.24)

$$\begin{aligned}V_h &= \mathcal{V}^{(s=0)}_{n/2-h,-n/2+h}\mathcal{V}'^{(s=2)}_{\frac{n+2}{2}-h,-\frac{n+2}{2}+h}\,,\\ J^{--}_{0L}J^{--}_{0R}V_h &= \mathcal{V}^{(s=-2)}_{n/2-h,-n/2+h}\mathcal{V}'^{(s=0)}_{\frac{n+2}{2}-h,-\frac{n+2}{2}+h}\,.\end{aligned} \tag{4.6}$$

The integrated vertex operators[16] take the form $G^-_{-1/2}V$. The G^- action is divided into two parts depending on whether it acts on the SL$(2,R)$ or SU(2) part. The former is given by

$$G^{-(sl)}_{-1/2}V_h = \mathcal{V}^{(s=2)}_{h-1,h-1}\mathcal{V}'^{(s=-2)}_{h,h+1}\,, \tag{4.7}$$

while the latter is

$$G^{-(su)}_{-1/2}V_h = \mathcal{V}^{(s=-2)}_{\frac{n}{2}-h,h-\frac{n}{2}+1}\mathcal{V}'^{(s=2)}_{\frac{n+2}{2}-h,-\frac{n+2}{2}+h}\,. \tag{4.8}$$

We also need the vertex operators in which G^- is replaced by $\tilde{G}^-$. Since V_h is chiral separately in both components it is very easy to find the action of $\tilde{G}^-_{-1/2}$

$$\tilde{G}^-_{-1/2}V_h = [J^{--}_0, G^+_{-1/2}]V_{\theta=0} = -G^+_{-1/2}J^{--}_0V_j\,. \tag{4.9}$$

[15]In terms of the low-energy effective action, these interactions schematically look like $\mathrm{Tr}[F^4B^{N-4}]$, where F is the six dimensional $\mathrm{U}(1)^n$ gauge field and B is the Higgs boson corresponding to the transverse motion of NS5-branes.

[16]We determined the normalization of these operators such that the cyclicity of amplitudes holds.

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This is divided into two parts: one obtained from the SU(2) action of $\tilde{G}$

$$\tilde{G}^{-(su)}_{-1/2} V_h = -\mathcal{V}^{(s=2)}_{h-1,h-2} \mathcal{V}'^{(s=-2)}_{h,h} \tag{4.10}$$

and the other one from the $\mathrm{SL}(2,R)$ action

$$\tilde{G}^{-(sl)}_{-1/2} V_h = -\mathcal{V}^{(s=-2)}_{\frac{n}{2}-h,-\frac{n}{2}+h} \mathcal{V}'^{(s=2)}_{\frac{n+2}{2}-h,-\frac{n+2}{2}-1+h} . \tag{4.11}$$

The correlators which involve $\tilde{G}^-$ can be treated in the $N = 2$ string equally well. In general if one has an amplitude with $L - 4$ G^- and $N - L$ $\tilde{G}^-$, then

$$\begin{aligned} A_{(N,L)} = \Big\langle & \phi^{(0,-1)}_{1,int} \phi^{(0,-1)}_{2}(z_2) \phi^{(-1,0)}_{3}(z_3) \phi^{(-1,0)}_{4}(z_4) \\ & \times \prod_{a=5}^{L} |PCO^-|^2 \phi^{(-1,0)}_{a,int} \prod_{a=L+1}^{N} |PCO^+|^2 \phi^{(0,-1)}_{a,int} \Big\rangle . \end{aligned} \tag{4.12}$$

4.2 Relation between SL(2, R) WZW model and bosonic Liouville theory

To evaluate (4.2), we need the N-point correlation functions in the supercoset SCFT (2.29). They are essentially reduced to the bosonic $\mathrm{SU}(2)_{n-2}$ and $\mathrm{SL}(2,R)_{n+2}$ WZW model as can be seen from the formulae (2.20) and (2.26). This is because the fermions (i.e. superpartners)in the supercoset are essentially free.

Even though there are no known general expressions for them except for the three point functions [36, 37], recently a remarkable relation has been uncovered [21 − 24] between correlation functions in the bosonic $\mathrm{SL}(2,R)_{n+2}$ WZW model[17] and those in the bosonic Liouville theory. In this subsection we review this relation for later convenience. In the recent paper [1], the computations of N-point functions in the $N = 2$ topological string on $\mathrm{SL}(2,R)/\mathrm{U}(1)$ WZW model have been done in order to find further evidence for the equivalence between the twisted coset with the level $n = 1$ or $n > 1$ and the $c = 1$ string [38] or the (non-minimal) $c < 1$ string [39] (see also the recent discussion [40]).

The bosonic Liouville field ϕ has background charge $Q = b + 1/b$ and the Liouville interaction is given by

$$\mathcal{L}_{int} = \mu \int d^2 z e^{2b\phi} . \tag{4.13}$$

In the mentioned relation, the bosonic $\mathrm{SL}(2,R)_{n+2}$ model is mapped to the bosonic Liouville theory with $b = \frac{1}{\sqrt{n}}$ and $\mu = \frac{b^2}{\pi^2}$. This model has the central charge $c = 1 + 6(n + 1)^2/n$. Notice that when n is integer this Liouville theory appears in the $(1, n)$ minimal bosonic string. The (spectral flowed) primary field $\Phi^w_{h,m,\bar{m}}$ is mapped to the primary $U_\gamma = e^{2\gamma\phi}$ in the Liouville CFT, where γ is defined by[18]

$$\gamma = b(1 - h) + \frac{1}{2b} = \frac{1}{\sqrt{n}}(1 - h) + \frac{\sqrt{n}}{2} . \tag{4.14}$$

[17] We assume the usual analytical continuation of the H_3^+ model to find results in the $\mathrm{SL}(2,R)$ WZW model.

[18] Notice that our definition of h is related to the ordinary spin j in [21], [22], [23], [24] via $j = -h$.

Then the explicit map of N-point functions in both theories is given by[19][21–24]

$$\left\langle \prod_{i=1}^{N} \Phi^{w_i}_{h_i,m_i,\bar{m}_i}(z_i) \right\rangle_{\sum_i w_i = r > 0}$$
$$= \frac{2\pi^{3-2N} b (c_{n+2})^r}{(N-r-2)!} \prod_{l=1}^{N} N_{h_l,m_l,\bar{m}_l} \cdot \delta^{(2)}\left(\sum_l m_l - \frac{n+2}{2} r \right)$$
$$\times \left| \prod_{l<l'} z_{ll'}^{\beta_{ll'}} \right|^2 \int \prod_{a=1}^{N-2-r} dy_a^2 \prod_{a<a'} |y_a - y_{a'}|^{n+2} \cdot \left| \prod_{l,a} (z_l - y_a)^{-\frac{n+2}{2}+m_l} \right|^2$$
$$\times \left\langle \prod_{l=1}^{N} U_{\alpha_l}(z_l) \prod_{a=1}^{N-2-r} U_{-\frac{1}{2b}}(y_a) \right\rangle , \tag{4.15}$$

where we defined $z_{ij} = z_i - z_j$ and

$$\begin{aligned} \beta_{ll'} &= \frac{n+2}{2} - \frac{n+2}{2} w_l w_{l'} + w_l m_{l'} + w_{l'} m_l - m_l - m_{l'} , \\ N_{h,m,\bar{m}} &= \frac{\Gamma(h-m)}{\Gamma(1+\bar{m}-h)} . \end{aligned} \tag{4.16}$$

Also c_{n+2} is a certain unknown constant which depends on n. A similar formula for total negative winding number $\sum_i w_i = -r < 0$ can be found by setting $m_i \to -m_i$.

Below we will be interested in the case $r = N-2$, where the maximal winding number violation occurs. Only in this case (and also in the minimally violated case), we do not have any insertions of the vertex $U_{-\frac{1}{2b}}(y_a)$ and get the following simple formula.[20]

$$\left\langle \prod_{i=1}^{N} \Phi^{w_i}_{h_i,m_i,\bar{m}_i}(z_i) \right\rangle_{\sum_i w_i = N-2} = 2\pi^{-1} b \cdot (c_{n+2}/\pi^2)^{N-2} \cdot \prod_{l=1}^{N} N_{h_l,m_l,\bar{m}_l} \tag{4.17}$$
$$\cdot \delta^{(2)}\left(\sum_l m_l - \frac{n+2}{2}(N-2) \right) \cdot \left| \prod_{l<l'} z_{ll'}^{\beta_{ll'}} \right|^2 \cdot \left\langle \prod_{l=1}^{N} U_{\alpha_l}(z_l) \right\rangle .$$

4.3 Emergence of $(1,n)$ minimal bosonic string

To analyze (4.2) we also need to express the N-point functions of the bosonic $\mathrm{SU}(2)_{n-2}$ WZW in terms of correlators in the minimal model. In order to establish this correspondence recall that the SU(2) algebra at the level $k = n-2$ is the same as the $\mathrm{SL}(2,R)$ algebra at the negative level $k = -n-2$. Employing this fact, we can regard the correlation functions of the SU(2) model as those in the $\mathrm{SL}(2,R)$ model.

[19] Our definition of the winding number w is opposite to the references mentioned. In terms of the primary in the $N=2$ coset, the quantum number s is related to w via (2.27) (2.19).

[20] This relation can be intuitively understood from the equivalence between the $N=2$ twisted coset $\mathrm{SL}(2,R)_{n+2}/\,\mathrm{U}(1)$ and the $c \le 1$ string as noted in [39, 1]. Consider the free field representation (Wakimoto representation) of $\mathrm{SL}(2,R)_{n+2}$ WZW model in terms of a bosonic scalar field with the background charge $Q = \frac{1}{\sqrt{n}}$ as well as the bosonic ghosts (β,γ). After the topological twist, the background charge of the scalar field becomes $Q = \sqrt{n} + \frac{1}{\sqrt{n}}$, which is the same as the one in the Liouville theory with $b = \frac{1}{\sqrt{n}}$. Then we can indeed confirm the equivalence in the free field representation as in [38, 39].

This leads to the relation between the bosonic $\mathrm{SU}(2)_{n-2}$ WZW model and the bosonic Liouville theory with the imaginary parameter $b = -\frac{i}{\sqrt{n}}$, which has the central charge $c = 1-6(n-1)^2/n$. The (spectral flowed) primary fields $\Psi^w_{j,m}$ are mapped to the $U_{\gamma'} = e^{2i\gamma'\varphi}$ in the Liouville theory, with

$$\gamma' = -\frac{1}{\sqrt{n}}(j+1) + \frac{\sqrt{n}}{2}\,. \tag{4.18}$$

The map between correlation functions can be obtained from (4.15), (4.18) by taking $n \to -n$ and $j \to -h$.

After the Wick rotation, this theory becomes the time-like Liouville theory [41, 42]. It has the same central charge as the minimal $(1,n)$ model. Thus we can expect that the correlation functions in this Liouville theory and the minimal $(1,n)$ model are the same. For (p,q) $(1 < p < q)$ models, this equivalence was checked in [43] by computing the three point functions.

The minimal $(1,n)$ model can not be regarded as a minimal model in a usual sense because its Kac table is empty (there are only $(p-1)(q-1)/2$ primaries in the Kac table of minimal (p,q) model). However, it is known that such a matter CFT coupled with the Liouville theory is a well-defined string theory and it plays an important role especially from the matrix model viewpoint (see e.g. the review [44]). In particular, one can deform the $(1,n)$ minimal string into the (p,n) minimal string as can be understood from the integrable hierarchy arguments.

There are infinitely many physical states in the $(1,n)$ minimal string. In the Coulomb gas representation they are given by

$$cT_{(r,s)} = cW_{(r,s)} \cdot e^{\frac{r+1-(s-1)n}{\sqrt{n}}\phi} \qquad (r = 1,2,\ldots,n-1, \quad s = 1,2,3,\ldots)\,. \tag{4.19}$$

The r-quantum number is truncated as usual by the screening charge, while s is unrestricted. (see [45, 46, 20] for more details). The operators $W_{(r,s)}$ are the primary fields in the $(1,n)$ model

$$W_{(r,s)} = e^{2i\alpha_{r,s}\varphi}, \qquad \alpha_{r,s} = -\frac{1-r}{2}\frac{1}{\sqrt{n}} + \frac{1-s}{2}\sqrt{n}\,. \tag{4.20}$$

The particular operators $T_{(r,1)}$ (i.e. $s = 1$) will play an important role in the later discussions. In the context of topological gravity (see the appendix B), they correspond to the matter chiral states while the others $s = 2,3,\cdots$ are the gravitational descendants. The lowest one $T_{(1,1)}$ is called the puncture operator.

If we combine results of the previous and present subsections, we can find an interesting connection between the supercoset (2.29) and the $(1,n)$ minimal string. Indeed we found that the $\mathrm{SL}(2,R)$ part is mapped to the bosonic Liouville theory ($c_L = 1+6(n+1)^2/n$), while the $\mathrm{SU}(2)$ part is mapped to the $(1,n)$ model ($c_m = 1-6(n-1)^2/n$). Together they give the matter part of the minimal $(1,n)$ model (note that $c_m + c_L = 26$). This motivates us to investigate the correlation functions further in order to see if those two theories are related.

4.4 Four point functions

Now we move back to the analysis of the scattering amplitudes (4.1). Let us begin with the four point functions. By applying the formula (4.15), we can rewrite the four point function for the vertices (4.5). Since the total winding number is $\sum_{i=1}^4 w_i (= -2\sum_i s_i) = 2$ in the SL$(2,R)$ sector, it is maximally winding number violating amplitude. On the other hand, in the SU(2) sector, it corresponds to minimal winding number violation. Then we can employ the simplified formula (4.18) and its counterpart on the SU(2) side.

In addition to the four point function in the bosonic Liouville and the $(1,n)$ model, we encounter some complicated factors of the form $\prod_{l<l'} |z_{ll'}^{\beta_{ll'}}|^2$ when we evaluate (4.2). First of all, such factors arise in the formula (4.15) in both SU(2) and SL(2) sectors. Also we have to take into account similar factors (2.22) and (2.28) , which arise when we relate the correlation functions of the $N=2$ cosets to the original WZW models via (2.20) and (2.26). Interestingly, we find that all these factors are almost canceled with each other, leaving us the simple factors $|z_1-z_2|^2|z_3-z_4|^2$. Combined with the similar factors in the original expression (4.2) we obtain $|z_2-z_3|^2|z_2-z_4|^2|z_3-z_4|^2$ in the end. Notice that this is the same as the familiar c-ghost correlation function $\langle c(z_2)\bar{c}(z_2)c(z_3)\bar{c}(z_3)c(z_4)\bar{c}(z_4)\rangle$.

In this way we can relate the four point functions in $N=4$ TST to the ones in $(1,n)$ minimal bosonic string (refer to [47], [48] for the correlation functions in minimal bosonic string) as follows

$$\begin{aligned} A_4 = & \ C\cdot\delta^{(4)}\left(\sum_{i=4}^{4} h_i - 2 - n\right)\cdot\int d^2 z_1 \langle c(z_2)\bar{c}(z_2)c(z_3)\bar{c}(z_3)c(z_4)\bar{c}(z_4)\rangle \\ & \times \left\langle T_{(r_1,1)}(z_1,\bar{z}_1)\ T_{(r_2,1)}(z_2,\bar{z}_2)\ T_{(r_3,1)}(z_3,\bar{z}_3)\ T_{(r_4,1)}(z_4,\bar{z}_4)\right\rangle , \end{aligned} \tag{4.21}$$

where the fields $T_{(r,1)}$ are the physical vertex operators in $(1,n)$ bosonic string, defined in (4.19), and they correspond to the operators $V_{h=1,3/2,\cdots,n/2}$ in (4.5) via the relation

$$r = n+1-2h \qquad (=1,2,\ldots,n-1)\,. \tag{4.22}$$

This relation between h and r comes from the maps (4.14) and (4.18). The factor C is an overall constant $C = \frac{-2i}{\pi^2 n}\cdot(\frac{c_{n+2}c_{-n+2}}{\pi^2})^2\left(\frac{\Gamma(0)}{\Gamma(1)}\right)^8$; we can absorb the divergent piece[21] $(\frac{\Gamma(0)^2}{\Gamma(1)^2})^N$ in the normalization of each vertex V_{h_i} in the $N=4$ TST.

We have found that the physical states $V_{h=1,3/2,\ldots,n/2}$ (3.12) in the $N=4$ topological string on ALE spaces are in one to one correspondence with the states $T_{(r=1,2,\ldots,n-1,\ s=1)}$ in the minimal $(1,n)$ string. It is also natural to expect that the other states $T_{(r,s>1)}$ may correspond to some other physical states in the $N=4$ topological string in a way similar to the gravitational descendants, though we will not discuss this issue in this paper.

Then the four point functions can be written simply as follows (up to an unimportant factor and the delta function in the (4.22))

$$A_4 = \langle T_{(r_1,1)}T_{(r_2,1)}T_{(r_3,1)}T_{(r_4,1)}\rangle_{(1,n)\text{string}}\,. \tag{4.23}$$

[21]We can also absorb the factor $c_{n+2}c_{-n+2}\pi^{-2}$ in the normalization. This is because the N-point functions always include the factor $(c_{n+2}c_{-n+2}\pi^{-2})^{N-2}$.

In addition we have the following constraint from the R-charge conservation (i.e. the δ-function in (4.21))

$$\sum_{i=1}^{4} h_i = 2+n, \qquad \left(\text{or} \quad \text{equivalently} \quad \sum_{i=1}^{4} r_i = 2n\right). \tag{4.24}$$

This constraint is not clear from the viewpoint of the $(1,n)$ minimal string.

On the other hand, if we apply the other equivalent representation (4.6), then we find another expression after a similar analysis

$$A_4 = \langle T_{(\tilde{r}_1,1)}\ T_{(\tilde{r}_2,1)}\ T_{(\tilde{r}_3,1)}\ T_{(\tilde{r}_4,1)} \rangle_{(1,n) string}\,, \tag{4.25}$$

with the same constraint (4.24). Here we defined the integer $\tilde{r}$ by

$$\tilde{r} \equiv n-r = 2h-1 \qquad (=1,2,\ldots,n-1)\,. \tag{4.26}$$

These two expressions (4.23) and (4.25) should be identical. This suggests that the theory has the $\mathbb{Z}_2$ global symmetry which replaces all of the operators $T_{(r,1)}$ with the $T_{(\tilde{r},1)}$. This symmetry is consistent with the four point function expression[22] conjectured in [13] from the duality between Heterotic string on T^4 and Type II string on K3

$$A_4 = \min\{r_i, n-r_i\}\,. \tag{4.27}$$

4.5 Five point functions

Now let us proceed to five point functions. In this case we can still rewrite the correlation functions in terms of those in the $(1,n)$ minimal string. We have two choices for the integrated operator corresponding to the action of G^- and $\tilde{G}^-$. We can again apply the formula (4.18) and simplify the total expression in the same way as in the four point function case. In the end, we obtain the following results classified into four cases

$$\begin{aligned}
A_5^{(1)} &= \left\langle \left(\int J^{--}_{0R} J^{--}_{0L} V_{h_1} \right) (J^{--}_{0R} J^{--}_{0L} V_{h_2}) \cdot V_{h_3} \cdot V_{h_4} \cdot \left(\int G^{-}_{-1/2R} G^{-}_{-1/2L} V_{h_5} \right) \right\rangle \\
&= \langle T_{(r_1,1)} T_{(r_2,1)} T_{(r_3,1)} T_{(r_4,1)} T_{(r_5,1)} \rangle_{(1,\text{n})\text{string}} \quad \text{when} \ \sum_i h_i = 2+3n/2\,, \\
A_5^{(2)} &= \left\langle \left(\int J^{--}_{0R} J^{--}_{0L} V_{h_1} \right) (J^{--}_{0R} J^{--}_{0L} V_{h_2}) \cdot V_{h_3} \cdot V_{h_4} \cdot \left(\int G^{-}_{-1/2R} G^{-}_{-1/2L} V_{h_5} \right) \right\rangle \\
&= \langle T_{(\tilde{r}_1,1)} T_{(\tilde{r}_2,1)} T_{(\tilde{r}_3,1)} T_{(\tilde{r}_4,1)} T_{(\tilde{r}_5,1)} \rangle_{(1,\text{n})\text{string}} \quad \text{when} \ \sum_i h_i = 2+n\,, \\
A_5^{(3)} &= \left\langle \left(\int J^{--}_{0R} J^{--}_{0L} V_{h_1} \right) (J^{--}_{0R} J^{--}_{0L} V_{h_2}) \cdot V_{h_3} \cdot V_{h_4} \cdot \left(\int \tilde{G}^{-}_{-1/2R} \tilde{G}^{-}_{-1/2L} V_{h_5} \right) \right\rangle
\end{aligned}$$

[22] It is intriguing to note that this expression coincides with the four point function [49] in the topological gravity with $(n-1)$-th minimal matter. The latter theory is usually associated with $(1,n+1)$ bosonic string without Liouville potential as reviewed in appendix B, instead of $(1,n)$ minimal bosonic string with Liouville wall. The possibility of the connection between the $N=4$ TST and the $(1,n+1)$ string has already been implied in [19] from the analysis of R-charge conservation. For more details see the appendix B.

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$$= \langle T_{(r_1,1)}T_{(r_2,1)}T_{(r_3,1)}T_{(r_4,1)}T_{(r_5,1)}\rangle_{(1,\mathrm{n})\mathrm{string}} \quad \text{when} \quad \sum_i h_i = 3 + 3n/2\,,$$

$$A_5^{(4)} = \left\langle \left(\int J_{0R}^{--} J_{0L}^{--} V_{h_1} \right) (J_{0R}^{--} J_{0L}^{--} V_{h_2}) \cdot V_{h_3} \cdot V_{h_4} \cdot \left(\int \tilde{G}^-_{-1/2R} \tilde{G}^-_{-1/2L} V_{h_5} \right) \right\rangle$$

$$= \langle T_{(\tilde{r}_1,1)}T_{(\tilde{r}_2,1)}T_{(\tilde{r}_3,1)}T_{(\tilde{r}_4,1)}T_{(\tilde{r}_5,1)}\rangle_{(1,\mathrm{n})\mathrm{string}} \quad \text{when} \quad \sum_i h_i = 3 + n\,, \tag{4.28}$$

where again we defined $\tilde{r} = n - r$. The above results for $A^{(1)}$ and $A^{(3)}$ are found by using the expressions (4.5) (4.7) (4.10), while those for $A^{(2)}$ and $A^{(4)}$ are done by applying (4.6) (4.8) (4.11).

4.6 $N(\geq 6)$-point functions

In the case of $N(\geq 6)$-point functions, the amplitudes (4.2) vanish unless the following R-charge conservation is satisfied

$$\sum_{i=1}^{N} h_i = n + 2 + l_1 + \frac{n}{2}\, l_2\,, \tag{4.29}$$

where the integers l_1 and l_2 takes the values $l_{1,2} = 0, 1, 2, \ldots, (N-4)$. If the numbers of insertions of $G^{-(sl)}$, $G^{-(su)}$, $\tilde{G}^{-(su)}$ and $\tilde{G}^{-(sl)}$ are denoted by a, b, c and d, then the integers l_1 and l_2 are written as $l_1 = N - 4 - a - b$ and $l_2 = a + c$.

We can again rewrite the N-point functions in terms of the correlation functions in the bosonic Liouville theory using (4.15). Unlike the four and five-point functions, it is not obvious how to reduce the general N-point functions to those in the $(1, n)$ string as long as we proceed just as before. This is because we do not have the maximally or minimally winding violation $\sum_i w_i = \pm(N-2)$ in general and we cannot eliminate the insertions of $V_{-\frac{1}{2b}}(y_a)$ in (4.15).

However, we can find that some of the amplitudes can be rewritten in terms of the $(1, n)$ string by applying (4.18). Consider the $N-$point function $A_N^{(1)}$ which includes $(N - 4 - l_1)$ $G^{-(sl)}V_h$ operators and l_1 $\tilde{G}^{-(su)}V_h$ ones, and also its dual amplitude $A_N^{(2)}$ obtained via $r \to \tilde{r}$. We can show[23] that they are the same as the $(1, n)$ minimal string amplitudes

$$A_N^{(1)} = \langle T_{(r_1,1)}T_{(r_2,1)} \ldots T_{(r_N,1)}\rangle_{(1,n)\mathrm{string}} \quad \text{when} \quad \sum_i h_i = (N-2)\frac{n}{2} + 2 + l_1\,,$$

$$A_N^{(2)} = \langle T_{(\tilde{r}_1,1)}T_{(\tilde{r}_2,1)} \ldots T_{(\tilde{r}_N,1)}\rangle_{(1,n)\mathrm{string}} \quad \text{when} \quad \sum_i h_i = n + 2 + l_1\,, \tag{4.30}$$

where $l_1 = 0, 1, \ldots, N-4$. These correspond to $l_2 = N - 4$ and $l_2 = 0$, respectively in (4.29).

Naively, amplitudes other than (4.30) do not seem to be reduced to the ones in the $(1, n)$ string. If we apply (4.15) and rewrite them, then they will include the extra insertions of integrated operators of the form $\int dy V_{-1/2b}(y)$. However, we cannot deny the possibility

[23]This consideration also determines the normalization of each vertex V_a in (4.2). We rescale the operators $V_{1,2,3,4}$ by multiplying the factor $\pi^2(c_{n+2}c_{-n+2})^{-1}\Gamma(0)^{-2}$. For the other integrated vertex operators, in addition to this factor, we also need to multiply i for each G^- action and $-i$ for $\tilde{G}^-$ action.

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that one can still relate the generic $N \geq 6$ amplitudes to the $(1,n)$ string amplitudes by performing the integral explicitly. This point needs future investigations and we will not pursue it here.

4.7 $N = 4$ topological string on A_1 space and matrix model for $(1,2)$ string

It is well-known that the $(1,n)$ minimal string is equivalent to the double scaled multi-matrix model. In the simplest case of the $(1,2)$ string ($c = -2$ string), it can also be thought of as the $k = 1$ case of the $(2, 2k-1)$ series. Then its matrix model dual is simply given by the gaussian one matrix model [9]. Since this matrix model can be solved exactly, we would like to use it to obtain the scattering amplitudes in $N = 4$ topological string on the A_1 type ALE space (Eguchi-Hanson space).

In this case the tree level correlation functions look like[24] [9, 50]

$$\left\langle \prod_{i=1}^{N} T_{(1,s_i)} \right\rangle = \tag{4.31}$$
$$= \mu^{\sum s_i - 2N+3} \frac{\Gamma(\sum_i s_i - N + 1)}{\Gamma(\sum_i s_i - 2N + 4)} \cdot \log \mu, \quad \left(\sum_i s_i - 2N + 3 \geq 0\right)$$
$$= \mu^{\sum s_i - 2N+3} \Gamma\left(\sum_i s_i - N + 1\right) \Gamma\left(2N - 3 - \sum_i s_i\right), \quad \left(\sum_i s_i - 2N + 3 < 0\right),$$

where we assumed $\sum_i s_i - N + 1 \geq 1$. Note that the terms with the positive powers of μ are accompanied with $\log \mu$, which is explained by the infinite volume in the Liouville direction. Thus they survive the double scaling limit along with the terms with negative powers of μ.

When we keep only the terms with $\log \mu$ singularity, the free energy $F/\log \mu = t^3/6 - 1/12$ and the scattering amplitudes (4.31) only include non-negative integer powers of the cosmological constant μ. In this sense the theory becomes topological and it is known to be equivalent to the pure topological gravity [51]. The similar model in the $(1,n)$ case is also known to be the same as the topological gravity coupled to the $(n-2)$-nd minimal topological matter [34, 52, 49, 53, 44]. Its matrix model dual is given by the generalized Kontsevich model. In the relation to the $N = 4$ topological string we also need to keep the terms with negative powers of μ so that the R-charge conservation (4.24) or (4.29) is satisfied.

Now let us consider the N-point functions of the $N = 4$ topological string, which only involve the integrated operators of the form $\int G_L^- G_R^- V_h$. Since in the $n = 2$ case, the SU(2) sector becomes trivial, we can only insert $G^{-(sl)}$ to obtain non-zero result. Hence all such tree level correlation functions can be rewritten in terms of the amplitudes of the $(1,2)$ minimal string just as we did for $A_N^{(1)}$ in (4.30). Now we can use the matrix model result (4.31) to compute these $N-$point correlation functions

$$A_N = \mu^{3-N} \cdot (N-4)!\,. \tag{4.32}$$

[24]Only in this subsection we recover the dependence on the cosmological constant to make things clear.

5. Discussion: toward matrix model for minimal $N = 2$ string

In this paper we studied the $N = 2$ minimal string and its connection to the $(1, n)$ minimal bosonic string. The $N = 2$ minimal string is equivalent to the $N = 4$ topological string near the ALE singularity. We concentrated on a particular set of physical states which correspond to the Kähler and complex structure deformations of the underlying geometry. Then we found one to one correspondence between them and physical states in the $(1, n)$ minimal string. It would be interesting to perform an exhaustive cohomology analysis and see if they are indeed equivalent at the free theory level.

To investigate this relation at the interaction level, we computed the closed string scattering amplitudes in the $N = 2$ minimal string or equivalently in the $N = 4$ topological string on ALE spaces. Indeed we found an intriguing connection to the $(1, n)$ minimal string. In particular, we showed that all four and five-point functions can be rewritten in terms of those of the $(1, n)$ minimal bosonic string. We were not able to match generic $N(\geq 6)$-point functions in the $N = 2$ string to that of the $(1, n)$ bosonic string although some classes of the higher point amplitudes do match.

These results suggest that these two string theories are closely related. On the other hand there also seem to be important distinctions. In particular we encountered the R-charge conservation like (4.24), which is not easy to understand from the viewpoint of $(1, n)$ string. Also methods used in this paper do not allow to match *generic* six and higher point functions. It would be interesting to understand whether these problems are artefact of the method or they indicate the non-equivalence of the two theories. Therefore it is very important to have future progress on both sides. Below we would like to discuss a possible matrix model dual for the $N = 2$ minimal string inspired by this connection.

5.1 ADE matrix model

It is very natural to expect that there exists a matrix model dual for the minimal $N = 2$ string or equivalently for the $N = 4$ topological string on ALE spaces, as is true both in the bosonic and type 0 minimal string. We expect that the dual matrix model is equivalent to an open string theory of infinitely many D0-branes in that string theory. It is dual to the closed string theory via the holography as was so in the two dimensional string theory [6]. Refer to [54, 55] for recent discussions of open-closed duality for $(1, n)$ minimal string in different contexts.

The relevant D-branes in our model should be the D2-branes wrapped on $n-1$ 2-cycles in the A_{n-1} ALE space. Indeed one can construct corresponding $N = 2$ supersymmetric (B-type) boundary states[25] [56]. The open string theory is expected to be described by a quiver-like theory with $n - 1$ nodes and $n - 2$ arrows.

We would like to point out that a possible matrix model dual of the $N = 2$ string on ADE ALE spaces may be given by the Kostov's ADE matrix model [57]. It is defined by

[25] Furthermore, one can show that the B or A-type boundary states preserve $N = 4$ boundary conformal symmetry in $\hat{c} = 2$ theory as is natural from the viewpoint of $N = 4$ topological string.

the following matrix model action in the A_{n-1} case

$$S = \sum_{a=1}^{n-1} \mathrm{Tr} \left[U(\Phi^{(a)}) + M^{(a)} \bar{M}^{(a)} \right] + \sum_{a=1}^{n-2} \left[\bar{M}^{(a)} \Phi^{(a)} M^{(a)} + \bar{M}^{(a)} M^{(a)} \Phi^{(a+1)} \right]. \qquad (5.1)$$

Generalization to D_n or E_n case is straightforward. In the A_{n-1} quiver, the eigenvalues of the adjacency matrix $C_{a,b}$ take the values $\beta_{(p)} = 2\cos(\pi p)$, where p runs the values $p = \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}$. Correspondingly, the matrix model is known to describe the non-critical bosonic string with the matter central charge $c = 1 - \frac{6p^2}{1-p}$ after a suitable double scaling limit. The choice of p corresponds to the choice of the background charge on the world-sheet. This fact becomes clearer in the equivalent RSOS model description [58]. When we choose the maximal value $p = \frac{n-1}{n}$ we have $c = 1 - 6\frac{(n-1)^2}{n}$, which is the same as the $(1, n)$ minimal bosonic string[26]. Notice that in the simplest case of $n = 2$, the matrix model (5.1) is reduced to the gaussian one matrix model, which is known to be equivalent to the $(1, 2)$ string.

The eigenvectors of $C_{a,b}$ are given by $v_p^{(a)} = \sqrt{\frac{2}{n}} \sin(\pi p a)$. For each $p = \frac{h}{n}$ ($h = 1, 2, \ldots, n-1$), we have a corresponding operator which may be identified with the closed string primary $T_{(r=h,1)}$ (4.19). The integer h is the discrete Fourier transformation with respect to the 'position' $a = 1, 2, .., n-1$. Notice also that the $\mathbb{Z}_2$ symmetry $T_{(r,1)} \leftrightarrow T_{(\tilde{r}=n-r,1)}$ mentioned in section 4.4 becomes obvious in this description, being identified with the $\mathbb{Z}_2$ reflection symmetry of A_{n-1} Dynkin diagram.

5.2 World-sheet discretization

Another way to find a matrix model dual to closed string is via worldsheet discretization. How should the candidate matrix model discretize the worldsheet of the $N = 4$ topological string embedded into the ALE space? Since the string theory is topological, we expect that the worldsheet is localized on the non-trivial two-cycles of the target space. Then the sigma model map from a Riemann surface Σ to the A_{n-1} ALE space *is now reduced to a map from* Σ *to the* $n-1$ *points*. These $n-1$ points specify which 2-cycle a point on the Riemann surface is situated at. Hence the worldsheet is divided into regions, and the adjacent regions are mapped into the adjacent two cycles.

The ADE matrix model (5.1) does precisely this as we explain below. When we pick a Feynman diagram for (5.1), we find that the chains of propagators of $M^{(a)}$ field behave as non-intersecting loops. They divide the whole net of the Feynman diagram into regions bounded by these loops (see the left figure in Fig. 1). Such a model is known as the loop gas model (or $O(n)$ model). By taking its dual lattice, it is also equivalent to the model called RSOS model [58].

From this viewpoint, we have infinitely many domains surrounded by the loops on random Riemann surfaces. We can assign one number a (called height variable) out of the $n-1$ integers $a = 1, 2, 3, \ldots, n-1$ to each domain. In the matrix model (5.1) language,

[26] We can get a similar range of matter central charge for affine $\hat{A}_{2n-1}$ quiver matrix model. We believe this corresponds to the non-minimal $(1, n)$ string which is equivalent to the $N = 2$ topological string on $\mathrm{SL}(2, R)/\mathrm{U}(1)$ [39].

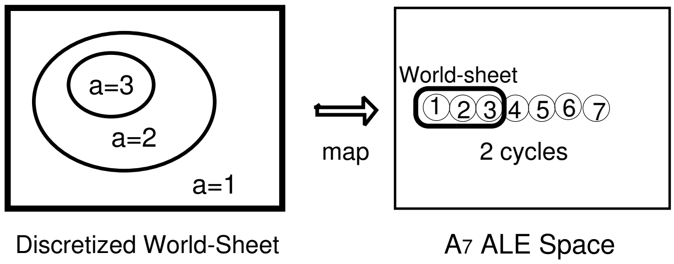

Figure 1: Discretization of world-sheet and the sigma-model map into A_7 ALE space.

this is the index a of the matrices $\Phi^{(a)}$. Then we have the requirement that the two domains which are adjacent should take the values a which are different from each other by ± 1 as is clear from the matrix model. Finally we assign a specific weight for a loop and sum over all such configurations. In this way we find that this model describes the map from a random surface to $n-1$ points (specified by the integer a) as we expected for the topological string on ALE spaces (see the right figure in Fig.1). Notice that this integer a describes the types of 2-cycles which D2-branes are wrapped on and thus this matches the above observation. The origin of $N=2$ world-sheet supersymmetry is not clear from this observation, unfortunately. This issue has not completely been understood even in the type 0 string theory, though there have been some progress [54]. We leave further analysis for future publications.

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A. $N=4$ topological string on $\mathbb{C}^2/\mathbb{Z}_n$

Here we study physical states of the $N=4$ topological string on $\mathbb{C}^2/\mathbb{Z}_n$ (see also [17] for $N=2$ string on $\mathbb{C}^2/\mathbb{Z}_2$). The world-sheet theory is described by the scalar fields $(X^1, X^2, \bar{X}^1, \bar{X}^2)$ and their superpartners $(\psi^1, \psi^2, \bar{\psi}^1, \bar{\psi}^2)$. The $\mathbb{Z}_n$ orbifold action is defined by $(X^1, X^2) \to (e^{2\pi i/n}X^1, e^{-2\pi i/n}X^2)$. The (left-moving) superconformal generators are

$$G^+ = \partial\bar{X}^1\psi^1 + \partial\bar{X}^2\psi^2, \qquad \tilde{G}^+ = \partial X^1\psi^2 - \partial X^2\psi^1\,. \tag{A.1}$$

In the $k-$th and $(n-k)$-th twisted sector of the bosonic part of $\mathbb{Z}_n$ orbifold, the ground states are represented by the twist operators $\sigma_\pm(z,\bar{z})$, which are defined by the OPEs $(i=1,2)$

$$\partial X^1(z)\sigma^1_+(0) \sim z^{-1+k/n}\tau^1_+, \qquad \partial\bar{X}^1(z)\sigma^1_+(0) \sim z^{-k/n}\tau^1_{+'}\,,$$

$$\bar{\partial} X^1(\bar{z})\sigma_+^1(0) \sim \bar{z}^{-k/n}\tilde{\tau}_+^1, \qquad \bar{\partial}\bar{X}^1(\bar{z})\sigma_+^1(0) \sim \bar{z}^{-1+k/n}\tilde{\tau}_{+'}^1,$$
$$\partial X^1(z)\sigma_-^1(0) \sim z^{-k/n}\tau_-^1, \qquad \partial\bar{X}^1(z)\sigma_-^1(0) \sim z^{-1+k/n}\tau_{-'}^1,$$
$$\bar{\partial} X^1(\bar{z})\sigma_-^1(0) \sim \bar{z}^{-1+k/n}\tilde{\tau}_-^1, \qquad \bar{\partial}\bar{X}^1(\bar{z})\sigma_-^1(0) \sim \bar{z}^{-k/n}\tilde{\tau}_{-'}^1. \tag{A.2}$$

The similar OPE between X^2 and $\sigma_\pm^2$ can be obtained by replacing k/n with $1-k/n$ in (A.2). Notice also that if we go around $\sigma_\pm^1$ once, we will get the twist by $e^{\pm 2\pi i k/n}$. Therefore, we are indeed considering the $k-$th and $(n-k)-$th twisted sector at the same time.

The condition (3.11) requires that the OPEs $G^+(z)V_{phys}(0)$, $\tilde{G}^+(z)\ V_{phys}(0)$, $G^+(\bar{z})$ $V_{phys}(0)$ and $\tilde{G}^+(\bar{z})V_{phys}(0)$ do not include the singularities z^{-n} or $\bar{z}^{-n}$ $(n \geq 1)$. Then we can find only one physical state in each $k-$th twisted sector

$$V_{phys}^{N=4TST(k)} = \sigma_+^1\sigma_+^2 e^{i\frac{k}{n}H_1} e^{i(1-\frac{k}{n})H_2} e^{i(1-\frac{k}{n})\tilde{H}_1} e^{i\frac{k}{n}\tilde{H}_2}, \tag{A.3}$$

where $H_{1,2}$ are the bosonizations of the left-moving fermions $\psi^{1,2}(z) = e^{iH_{1,2}}$ (OPEs are $H_i(z)H_j(0) \sim -\delta_{ij}\log z$). $\tilde{H}_{1,2}$ are the ones for the right-moving part.

Indeed it has the R-charges $J_3 = \tilde{J}_3 = 1$ and the conformal dimension $\Delta = 1/2$ before twisting (notice that $\Delta(\sigma_\pm) = -\frac{1}{2}(\frac{k}{n})^2 + \frac{1}{2}(\frac{k}{n})$). This operator (A.3) is $(chiral, chiral)$ primary in the $N=2$ SCFT and also belongs to the $k-$th twisted sector. By replacing $\sigma_+^{1,2}$ with $\sigma_-^{1,2}$, we find the physical state for $(n-k)-$th twisted sector. In this way we found $n-1$ twisted sector physical states. Geometrically they nicely correspond to the $n-1$ blowing up modes in the A_{n-1} ALE space. This number of physical states agrees with the singular ALE model studied in section 2, as we expected because these two different backgrounds should correspond to two different points in the moduli space of A_{n-1} ALE space. Note also that in this model we do not have the states written as $G_{-1/2}^+V$ opposite to the singular ALE case. Equivalently there are no $(-1,-1)$ picture physical states in this $N=2$ string except the trivial one.

B. Topological gravity

Consider the twisted $p-$th $N=2$ minimal model ($c = \frac{3p}{p+2}$) corresponding to the LG potential $W = X^{p+2}$ [34]. We can use the twisted $N=2$ minimal model as a matter theory to define the topological gravity [49, 59, 53] (see also the excellent review [44] on these matters). The most important physical states are chiral primaries and can be expressed as $\phi^{(m)} \sim X^m$ $(m = 0, 1, \ldots, p)$. The lowest operator $\phi^{(0)}$ is called the puncture operator. There are other types of physical states called gravitational descendants, which we will not consider here.

The state $\phi^{(m)}$ has the R-charge $q = \frac{m}{p+2}$. The $N-$point amplitudes are defined by

$$A_N = \left\langle \phi^{(i_1)}\phi^{(i_2)}\phi^{(i_3)} \prod_{a=4}^{N} \int G_L^- G_R^- \phi^{(i_a)} \right\rangle. \tag{B.1}$$

The R-charge (or ghost charge) conservation leads to

$$\sum_{i=1}^{N} \frac{m_i}{p+2} - (N-3) = \hat{c} = 1 - \frac{2}{p+2}. \tag{B.2}$$

As we have mentioned in section 4, this theory of topological gravity is known to be equivalent to the $(1,n)$ minimal bosonic string [52, 45]. In this context, the condition (B.1) can be understood by considering a minimal $(1,n)$ bosonic string *without Liouville potential* (but with the screening charge in the minimal matter sector). We identify $\phi^{(m)}$ with the tachyon operator

$$T_{(r=n-1-m,s=1)} = W_{(r,1)} \cdot e^{\frac{(n-m)}{\sqrt{n}}\phi}\,, \tag{B.3}$$

setting $n = p+2$. Indeed the momentum conservation in the Liouville ϕ direction coincides with (B.2). Furthermore, we can even prove that all tree level N-point functions of the $N=2$ twisted minimal model are the same as those in the $(1,n)$ minimal string using the relation (4.15). Since this computation is analogous to what we did in section 4 for the $\mathrm{SU}(2)/\,\mathrm{U}(1)$ sector, we omit the details.

In topological gravity, we usually talk about correlation functions expanding around the point where the cosmological constant is vanishing. Three and four point function [59], [49] are given by

$$\begin{aligned} \langle \phi^{m_1}\phi^{m_2}\phi^{m_3}\rangle &= \delta_{m_1+m_2+m_3,p}\,, \\ \langle \phi^{m_1}\phi^{m_2}\phi^{m_3}\phi^{m_4}\rangle &= \frac{1}{p+2}\min(m_i, p+1-m_i)\cdot \delta_{m_1+m_2+m_3+m_4,2p+2}\,. \end{aligned} \tag{B.4}$$

The delta-function in three point function comes from the R-charge conservation as already mentioned.

Finally, let us compare this with $N=4$ TST. Looking at the SU(2) part in our case, let us identify[27] $r = n-m$ assuming $p = n-1$. The state $m=0$ is almost trivial as it vanishes when acted upon by G^- and thus it is plausible that it is absent in $N=4$ TST. Then (B.2) can be rewritten as $\sum_{i=1}^{N} r_i = 2n+4-N$. This agrees with the R-charge constraint of the four point function $A^{(4)}$ and the special amplitudes $A^{(1)}_N$ at $l_1 = N-4$ in (4.24) (4.30). Also the four point function itself agrees (up to a constant) with the four point function (B.4) in $N=4$ TST obtained from the Het/TypeII duality setting $n = p+1$ again. Even though it is possible that this is just a coincidence, it would be interesting to see if this is indeed true in general amplitudes.

C. Closed string partition functions

Here we compute the partition function for the $N=2$ minimal string. Define the NS-sector N=2 character for the $\hat{c}=2$ matter N=2 SCFT as usual

$$Z_{\hat{c}=2}(\tau,z) = \mathrm{Tr}\left[q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}y^{J_0}\bar{y}^{\bar{J}_0}\right], \tag{C.1}$$

where $q = e^{2\pi\tau}$ and $y = e^{2\pi i z}$. The expression of the $N=2$ string partition function is given by [28], [19]

$$Z_{N=2} = \int \frac{d\tau d\bar{\tau}}{\tau_2}\int dz d\bar{z}|Z_{\hat{c}=2}(\tau,z)|^2|Z_{gh}(\tau,z)|^2\,, \tag{C.2}$$

where $(\tau,\bar{\tau})$ denotes the ordinary torus moduli and $(z,\bar{z})$ denote the U(1) gauge field moduli peculiar to $N=2$ string; we parameterized $z = \theta_1 + \tau\theta_2$ assuming $0 \le \theta_{1,2} < 1$.

[27]Notice that this is different from more standard one (B.3).

The part $Z_{gh}(\tau, z)$ represents the ghost partition function and it is explicitly expressed as

$$Z_{gh}(z|\tau) = \frac{\eta(\tau)^6}{\theta_3(z|\tau)^2}\,. \tag{C.3}$$

Let us proceed in our specific example (2.29), taking into account the continuous modes and not the discrete states for simplicity. We expect this is enough if we are interested in the bulk terms which behave like $\sim \log \mu_{sl}$, where μ_{sl} is the $N = 2$ cosmological constant as we usually do in the bosonic or type 0 non-critical string to compare matrix models.

The final result reads

$$Z_{N=2} = c \cdot \log \mu_{sl} \int \frac{d\tau d\bar{\tau}}{(\tau_2)^2} \int_0^1 d\theta_1 d\theta_2 \ (\tau_2)^{3/2} |\eta(\tau)|^6 \sum_{l,l'} N_{l,l'} \chi^{(l)}(\tau, 0) \chi^{(l')}(\bar{\tau}, 0)\,. \tag{C.4}$$

$\chi^{(l)}$ is the spin $j = l/2$ character of $\mathrm{SU}(2)_{n-2}$ WZW model. The $N_{l,l'}$ represents the multiplicity of primaries and it is well-know that it has the ADE classification corresponding to the ADE classification of ALE spaces [19, 14, 60]. c is a computable constant if we fix normalization. This expression (C.4) is manifestly $N = 2$ modular invariant. Notice that the partition function in the end does not depend on z as was also true in the $N = 2$ string on $\mathbb{R}^{2,2}$.

Even though this expression (C.4) is clear from the equivalent CHS geometry $\mathbb{R}_\phi \times \mathrm{SU}(2)_{n-2}$, it is instructive to derive it from the viewpoint of the $N = 2$ minimal model coupled to the $N = 2$ Liouville theory. We write the bosonic fields in the $N = 2$ Liouville theory as (Y, ϕ). The matter partition function (C.1) is divided into the Liouville part and the other contributions $Z_{\hat{c}=2} = Z_L Z_{others}$. The former is simply given by

$$Z_\phi = \log \mu_{sl} \cdot \frac{1}{\sqrt{\tau_2} |\eta(\tau)|^2}\,. \tag{C.5}$$

By imposing that the R-charge is integral, we have $m/N - Qp \in Z$, where (l, m) is the primary (2.4) of $N = 2$ minimal model and p is the $Y-$momentum. This projection is equivalent to the $\mathbb{Z}_n$ orbifold in (2.29). Then we find

$$\begin{aligned} Z^{(l)}_{others} &= \sum_{l=0}^{N-2} \sum_{m \in Z_{2N}} \frac{\theta_3(z|\tau)}{\eta(\tau)} ch^{NS}_{lm}(\tau, z) \frac{\Theta_{m,N}(\tau, -2z/N) + \Theta_{m+N,N}(\tau, -2z/N)}{\eta(\tau)} \\ &= \sum_{l=0}^{N-2} \frac{\theta_3(z|\tau)}{\eta(\tau)^2} \chi^{(l)}(\tau, 0)\,, \end{aligned} \tag{C.6}$$

where we adopt the convention and applied an important identity in [60]. Each three factors in the second expression in (C.6) are the contributions from the fermions in the $N = 2$ Liouville, the $N = 2$ minimal model and the Y boson, respectively.

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